

Densities of 3-vertex graphs*

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Abstract

Let $d_i(G)$ be the density of the 3-vertex i -edge graph in a graph G , i.e., the probability that three random vertices induce a subgraph with i edges. Let $S \subseteq \mathbb{R}^4$ be the set of all quadruples (d_0, d_1, d_2, d_3) that are arbitrary close to 3-vertex graph densities in arbitrary large graphs. Huang, Linial, Naves, Peled and Sudakov have recently determined the projection of the set S to the (d_0, d_3) plane. We determine the projection of the set S to all the remaining planes.

1 Introduction

Many problems in graph theory relate to understanding possible combinations of subgraph densities in graphs. Indeed, the study of possible subgraph densities

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forms a very important area of extremal graph theory, which contains many classical results but which is also full of hard and challenging problems. The classical results include, e.g., Turán's Theorem [32] determining the maximum edge density in K_r -free graphs, Goodman's Bound [9] relating the densities of $\overline{K_3}$ and K_3 , and Kruskal-Katona Theorem [15, 18]. On the other hand, one of the recent breakthroughs in extremal graph theory was the description of possible densities of complete graphs in graphs with a given edge density, which is given in the exciting work of Razborov [27], Nikiforov [22] and Reiher [28].

While problems related to possible densities of small graphs may look innocent at the first sight, they can become incredibly challenging. For example, determining the minimum possible sum of densities of $\overline{K_4}$ and K_4 is a well-known problem in graph theory, which is open for more than five decades. Erdős [6] conjectured this minimum to be $1/32$, which was disproved by Thomasson [30], who constructed graphs with the sum of the two densities below $1/32$. However, despite extensive subsequent work on the problem, e.g., [8, 29, 31, 33], there is not even a construction that is believed to provide the tight bound for this problem. Hence, determining all possible densities of $\overline{K_4}$ and K_4 look completely hopeless. On the other hand, Huang, Linial, Naves, Peled and Sudakov [13], building on their results from [14], determined possible densities of $\overline{K_3}$ and K_3 . We contribute to this line of research by completing the description of possible densities of all pairs of 3-vertex graphs. Our results are encouraging to make an attempt to describe all possible combinations of 3-vertex graph densities, which would imply the earlier mentioned result of Razborov [27] on the minimum triangle density in a graph with a given edge density.

To state our results precisely, we need several definitions. The density $d(H, G)$ of a k -vertex graph H in a graph G is the probability that k randomly chosen vertices of G induce a subgraph isomorphic to H . We will be interested in densities of 3-vertex graphs. There are four 3-vertex graphs: the triangle K_3 , the cherry $K_{1,2}$, the co-cherry $\overline{K_{1,2}}$ and the co-triangle $\overline{K_3}$; let H_k , $k = 0, 1, 2, 3$, be the 3-vertex graph with k edges. Further, let $S \subseteq \mathbb{R}^4$ be the set of all quadruples (d_0, d_1, d_2, d_3) such that for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, there exists a graph G with at least n vertices such that the density of H_k in G differs from d_k by at most ε for $k = 0, 1, 2, 3$. This is equivalent to saying that $(d_0, d_1, d_2, d_3) \in S$ if and only if there exists a sequence of graphs such that their number of vertices tends to infinity and the density of H_k in the graphs forming the sequence converges to d_k for every $k = 0, 1, 2, 3$. In [13], the set S is referred to as the set of 3-local profiles of arbitrary large graphs. There is also an alternative description of the set S using the theory of graph limits, which we present in Section 2.

Let S_{ij} be the projection of the set S to the plane of the i -th and j -th coordinate. Huang, Linial, Naves, Peled and Sudakov [13], building on their results from [14], determined the projection S_{03} of the set S . In particular, they showed that $(d_0, d_3) \in S_{03}$ if and only if $d_0 \geq 0$, $d_3 \geq 0$, $d_0 + d_3 \geq 1/4$ (this inequality is

equivalent to Goodman's bound), and

$$d_3 \leq \max\{(1 - d_0^{1/3})^3 + 3d_0^{1/3}(1 - d_0^{1/3})^2, (1 - \alpha)^3\}$$

where α is the unique root in $[0, 1]$ of the equation $\alpha^3 + 3\alpha^2(1 - \alpha) = d_0$. The set S_{03} is visualized in Figure 1. The upper curve in Figure 1 corresponds to densities of co-triangles and triangles in the graph consisting of a complete graph and isolated vertices or in the complement of this graph.

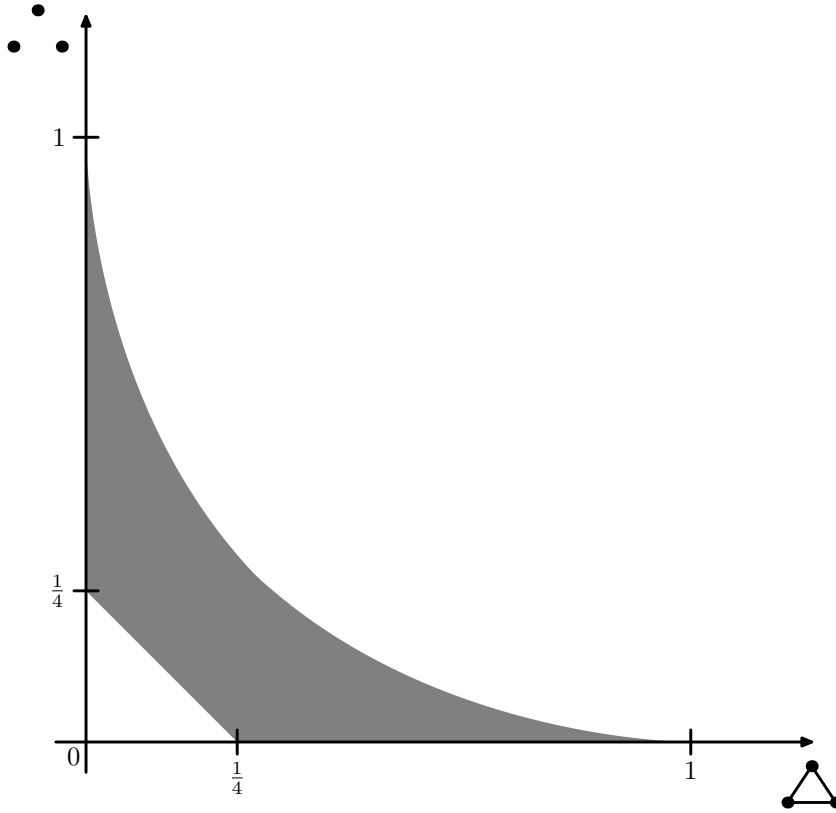


Figure 1: Possible densities of triangles and co-triangle in graphs.

In this paper, we determine the projections S_{ij} for all the other pairs of i and j . By considering the complements of graphs, it is easy to see that the projections S_{xy} and $S_{(3-x)(3-y)}$ are the same. Hence, it is enough to consider the following three projections only: S_{12} , S_{13} , and S_{23} , which we consider separately in Sections 4–6. While determining the projections S_{12} and S_{23} turned out to be relatively straightforward, the projection S_{13} , which we consider in Section 4, was significantly more difficult to describe. We would like to stress that although some of the proofs in Section 4 use the flag algebra method, all our proofs are computer-free and the method presents for us a very convenient way of formulating our arguments. In the concluding Section 7, we briefly discuss the possible structure of graphs with densities on the boundaries of the projections.

2 Graph limits and flag algebras

In this section, we briefly introduce the theory of graph limits and the flag algebra method. The flag algebra method can be presented independently of the theory of graph limits but since we will only apply the flag algebra method in the limit setting, we find it more convenient to introduce it using some graph limit notation. We also deal here only with limits of dense graphs, which we need in this paper, and refer the reader to a recent monograph by Lovász [20] for a more detailed exposition of graph limits.

Recall that the *density* $d(H, G)$ of a k -vertex graph H in a graph G is the probability that a randomly chosen subset of k vertices of G induces a subgraph isomorphic to H . A sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is *convergent* if the sequence of densities $d(H, G_n)$ converges for every graph H . In what follows, we will always assume that if $(G_n)_{n \in \mathbb{N}}$ is a convergent sequence, then the number of vertices of the graphs in the sequence tends to infinity.

Convergent sequences of graphs can be represented by an analytic object called a graphon; A *graphon* is a symmetric measurable function $W : [0, 1]^2 \rightarrow [0, 1]$, where symmetric stands for the property that $W(x, y) = W(y, x)$ for all $x, y \in [0, 1]$. We next define a density of a k -vertex graph H in a graphon W as follows. A k -vertex W -random graph is the random graph obtained by sampling k points x_1, \dots, x_n independently and uniformly in the unit interval $[0, 1]$ and joining the i -th vertex and the j -th vertex of the graph by an edge with probability $W(x_i, x_j)$. The *density* of a k -vertex graph H in W is the probability that a k -vertex W -random graph is isomorphic to H ; this density is denoted by $d(H, W)$. To facilitate reading we often use $d(H, W)$ where H is a drawing of the graph. We will sometimes refer to the elements of $[0, 1]$ as to the vertices of W , and we will say that the *degree* $d_W(x)$ of $x \in [0, 1]$ is the integral

$$\int_{[0,1]} W(x, y) \, dy,$$

which is the expectation of the fraction of the vertices in a W -random graph adjacent to a vertex associated with x .

A graphon W is a *limit* of a convergent sequence $(G_n)_{n \in \mathbb{N}}$ of graphs if the density $d(H, W)$ is equal to the limit of the densities $d(H, G_n)$ for every graph H . It is known that every convergent sequence of graphs has a limit [21], and this limit is unique up to certain measure preserving transformations [5]. In what follows, we use $|X|$ to denote the measure of a subset $X \subseteq [0, 1]$. If W is a graphon and A a non-null subset of $[0, 1]$, we define a graphon $W[A]$ *induced* by A as follows: fix any mapping $\varphi : [0, 1] \rightarrow A$ such that $|X| = |\varphi^{-1}(X)| \cdot |A|$ for every measurable $X \subseteq A$ and set $W[A](x, y) = W(\varphi(x), \varphi(y))$.

A *component* of a graphon W is a non-null subset $A \subseteq [0, 1]$ such that W is equal to zero almost everywhere on $A \times \overline{A}$ and there is no subset B of A such that both B and $A \setminus B$ are non-null and W is equal to zero almost everywhere

on $B \times (A \setminus B)$. It is not hard to show that for every graphon W , there is a countable (possibly finite) collection $(C_j)_{j \in J}$ of disjoint subsets of $[0, 1]$ such that each C_j is a component of W and W is zero almost everywhere outside $\cup_{j \in J} C_j^2$. Note that the set $[0, 1] \setminus \cup_{j \in J} C_j$ may be non-null.

The theory of graph limits can be used to present an alternative definition of the set S . A point (d_0, d_1, d_2, d_3) belongs to the set S if and only if there exists a graphon W such that $d(H_i, W) = d_i$ for $i = 0, 1, 2, 3$. Since the numbers d_0, d_1, d_2 and d_3 present the probabilities of sampling 3-vertex W -random graphs, the following becomes obvious.

Proposition 1. *If $(d_0, d_1, d_2, d_3) \in S$, then $d_0 + d_1 + d_2 + d_3 = 1$.*

Since the edge density of a graph is the average edge density in its 3-vertex subgraphs, we also get the following.

Proposition 2. *Let $(G_n)_{n \in \mathbb{N}}$ be a convergent sequence of graphs, and let d_e be its limit edge density and d_k the limit density of the graph H_k . It holds that $d_e = \frac{d_1 + 2d_2 + 3d_3}{3}$.*

We now introduce basic concepts related to the flag algebra method, which was developed by Razborov [25]. The method has become a popular tool in extremal combinatorics, see, e.g., [1–4, 10–12, 16, 17, 23, 24, 26, 27], and led to solving many long standing open problems in the area. In our exposition, we focus on presenting the main concepts only, and refer the reader for a more detailed exposition to, e.g., the original paper of Razborov [25].

Let \mathcal{F} be a set of finite formal linear combinations of graphs. We represent elements of \mathcal{F} as formal linear combinations of drawings of the corresponding graphs, e.g., $\mathcal{J} - \frac{1}{2} \mathcal{J} \bullet \mathcal{J}$. For a graphon, let $h_W : \mathcal{F} \rightarrow \mathbb{R}$ be the mapping that

$$h_W \left(\sum_{j \in J} \alpha_j G_j \right) = \sum_{j \in J} \alpha_j d(G_j, W) .$$

The mapping h_W respects both the addition of elements of \mathcal{F} and the multiplication by a scalar. Razborov [25] showed that it is possible to define a multiplication of the elements of \mathcal{F} that the mapping also respects this operation, i.e., $h_W(x \times y) = h_W(x)h_W(y)$ for all $x, y \in \mathcal{F}$. To keep our notation simple, we will occasionally write $d(x, W)$ instead of $h_W(x)$.

Suppose that $x, y \in \mathcal{F}$. We write that $x = y$ if $h_W(x) = h_W(y)$; if a graphon W is not specified, we mean that the equality holds for all graphons. For example, Proposition 2 can be rewritten as $\mathcal{J} = \frac{1}{3} \mathcal{J} \bullet \mathcal{J} + \frac{2}{3} \mathcal{J} \mathcal{J} + \mathcal{J} \mathcal{J} \mathcal{J}$. Likewise, we write $x \leq y$ if $h_W(x) \leq h_W(y)$. In particular, $0 \leq y$ if $h_W(y)$ is non-negative.

We now extend the just introduced concepts to rooted graphs. Let H be a graph that has k vertices and these are labelled with integers $1, \dots, k$. An H -rooted graph is a graph with k of its vertices labelled with $1, \dots, k$ in such a

way that the labelled vertices induce a copy of H in a way that preserves the labels. Let \mathcal{F}^H be the set of formal linear combinations of H -rooted graphs. In the analogy to \mathcal{F} , we depict elements of \mathcal{F}^H as linear combinations of drawing of H -rooted graphs where the roots are depicted with empty circles. If H has two or more vertices, all the elements of the sum have the copy of H depicted in the same way. For example, we will write $\mathcal{J} - \frac{1}{2} \circ \bullet$ or $\circ \circ + \mathcal{J} \circ \circ$.

Let $z_1, \dots, z_k \in [0, 1]$. We define a mapping $h_{W, z_1, \dots, z_k} : \mathcal{F}^H \rightarrow \mathbb{R}$ as follows. If G is an H -rooted graph with n unlabeled vertices, then $h_{W, z_1, \dots, z_k}(G)$ is the probability that a W -random graph is G conditioned on that the first k of the vertices $x_1, \dots, x_{k+n} \in [0, 1]$ being z_1, \dots, z_k and on that they induce a copy of H preserving the labels, i.e., the vertex $x_i = z_i$ is labelled with i . Note that the mapping h_{W, z_1, \dots, z_k} might not be defined for certain k -tuples z_1, \dots, z_k . We extend the mapping h_{W, z_1, \dots, z_k} by linearity to the whole set \mathcal{F}^H . Again, one can define the multiplication on \mathcal{F}^H in a way that the mapping respects the multiplication.

In the analogy with our earlier notation, we write $x = y$, $x \leq y$ and $0 \leq y$ for $x, y \in \mathcal{F}^H$ if the (in)equality holds for almost every choice of the roots z_1, \dots, z_k for which the mapping h_{W, z_1, \dots, z_k} is well-defined. It is possible to define a mapping $\llbracket \cdot \rrbracket_H$ from \mathcal{F}^H to \mathcal{F} such that the following holds for every $x \in \mathcal{F}^H$ and every graphon W

$$\mathbb{E}_{z_1, \dots, z_k} h_{W, z_1, \dots, z_k}(x) = h_W(\llbracket x \rrbracket_H),$$

where the expectation is taken with respect to the probability with the density function proportional to the probability that z_1, \dots, z_k induce a copy of H . If the graph H is clear from the context, we write $\llbracket x \rrbracket$ instead of $\llbracket x \rrbracket_H$. Since the square of any number of non-negative, the following proposition easily follows.

Proposition 3. *Let H be a labeled graph. For every $x \in \mathcal{F}^H$, it holds that $0 \leq \llbracket x^2 \rrbracket$.*

We finish this section with a simple example. Suppose that H is the single-vertex graph with its only vertex labelled with 1. If W is a graphon and $z \in [0, 1]$, then

$$h_{W, z}(\mathcal{J}) = \int_{0,1} W(z, x) dx.$$

For example, if W is the graphon that is equal to one on $[0, 1/3]^2 \cup (1/3, 1]^2$ and to zero elsewhere, then $h_{W, z}(\mathcal{J}) = 1/3$ if $z \in [0, 1/3)$, and $h_{W, z}(\mathcal{J}) = 2/3$, otherwise. Note that $\mathbb{E}_z h_{W, z}(\mathcal{J}) = h_W(\mathcal{J}) = 5/9$; we remark that it holds that $\llbracket \mathcal{J} \rrbracket = \mathcal{J}$.

3 Triangle density

In this section, we briefly recall some results on the minimum triangle density in large graphs with bounds on their minimum edge density. Perhaps, the oldest

bound of this type is the bound of Goodman [9], which can be written using the flag algebra language as

$$\blacktriangleright(2\blacktriangleright - 1) \leq \blacktriangle.$$

It has been a long-standing open problem to determine the optimum function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that every graphon with edge density being d_e has the triangle density at least $g(d_e)$. The value of the function was known for $d_e \leq 2/3$ due to work of Fisher [7] until Razborov [27] has solved this problem using his flag algebra method.

To state Razborov's result, we define the function $g_R : [0, 1] \rightarrow [0, 1]$ as follows. Set $g_R(d_e) = 0$ if $d_e \in [0, 1/2)$ and $g_R(1) = 1$. If $d_e \in [1/2, 1)$, let k be the smallest integer such that $d_e \leq 1 - 1/k$. Let z be the unique integer in the interval $(0, 1/k]$ such that

$$d_e = 1 - (k - 1) \cdot \left(\frac{1 - z}{k - 1} \right)^2 - z^2 = \frac{(1 - z)(kz + k - 2)}{k - 1},$$

and define

$$\begin{aligned} g_R(d_e) &= 6 \binom{k-1}{3} \left(\frac{1-z}{k-1} \right)^3 + 6 \binom{k-1}{2} z \left(\frac{1-z}{k-1} \right)^2 \\ &= \frac{(1-z)^2(k-2)(2zk + k - 3)}{(k-1)^2}. \end{aligned}$$

Note that $g_R(d_e)$ is the asymptotic triangle density in a complete k -partite graph such that one of its parts contain the fraction z of its vertices and the remaining $k - 1$ parts have the same size. Note that $g_R(d_e) = 0$ if $d_e \in [0, 1/2]$ and

$$g_R(d_e) = \frac{(1 - \sqrt{4 - 6d_e})(2 + \sqrt{4 - 6d_e})^2}{18} \quad (1)$$

for $d_e \in [1/2, 2/3]$. Razborov's result is equivalent to the following.

Theorem 4 (Razborov [27]). *If W is a graphon with edge density d_e , then the triangle density of W is at least $g_R(d_e)$. In particular, if $(d_0, d_1, d_2, d_3) \in S$, then $d_3 \geq g_R((d_1 + 2d_2 + 3d_3)/3)$.*

The set of feasible edge and triangle densities is depicted in Figure 2.

Pikhurko and Razborov [23] described structure of large graphs with edge density d_e and triangle density $g_R(d_e)$. We state their result in the language of graphons. Let d_e be in $[1/2, 1)$. Let k be the smallest integer such that $d_e \leq 1 - 1/k$, and let z be as in the definition of the function g_R . Further, denote $C_i = \left[\frac{(i-1)(1-z)}{k-1}, \frac{i(1-z)}{k-1} \right)$ for $i = 1, \dots, k-2$ and $C_{k-1} = \left[\frac{(k-2)(1-z)}{k-1}, 1 \right]$. Define a graphon W such that $W(x, y) = 0$ if $(x, y) \in C_i^2$ for $i = 1, \dots, k-2$, $W(x, y) = 1$

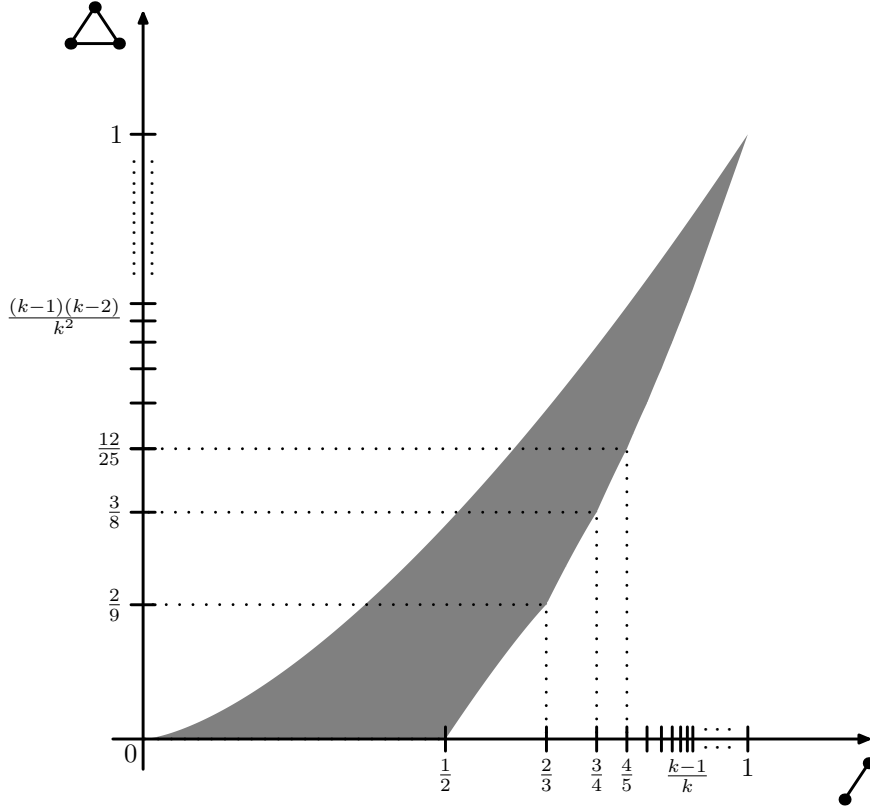


Figure 2: Possible densities of edges and triangles in graphs.

if $(x, y) \in C_i \times C_j$ for $i \neq j$, and $W[C_{k-1}]$ is a graphon with the zero triangle density, and the edge density equal to

$$\frac{2z(k-1)(1-z)}{(kz-2z+1)^2},$$

i.e., the edge density of the complete bipartite graph with parts containing the fractions of $\frac{(1-z)/(k-1)}{z+(1-z)/(k-1)}$ and $\frac{z}{z+(1-z)/(k-1)}$ of vertices. For any choice of $W[C_{k-1}]$, the graphon W has the edge density equal to d_e and the triangle density equal to $g_R(d_e)$. Pikhurko and Razborov [23] showed that if $(G_n)_{n \in \mathbb{N}}$ is a convergent sequence of graphs such that its limit edge density is $d_e \in [1/2, 1)$ and its triangle density is $g_R(d_e)$, then its limit is one of the graphons W defined above.

A stronger lower bound on the number of triangles can be shown assuming that every vertex is adjacent to at least the fraction $d \in [0, 1]$ of all vertices. Lo [19] proved tight structural results if $d \in [0, 3/4]$. We state his result in the complementary form that we apply later in our considerations.

Theorem 5 (Lo [19]). *Let $d \in [1/4, 1]$, and let W be a graphon that minimizes $d(\bullet, \bullet, W)$ subject to that $d_W(x) \leq d$ for almost every $x \in [0, 1]$. It holds that $d_W(x) = d$ for almost every $x \in [0, 1]$ and*

- if $d \in (1/2, 1]$, then graphon W has a single component of measure one and $d(\bullet\bullet, W) = 0$,
- if $d \in (1/3, 1/2]$, then W has two components C_1 and C_2 of measures $|C_1| = d$ and $|C_2| = 1 - d$, W is equal to one almost everywhere on C_1^2 and $d(\bullet\bullet, W[C_2]) = 0$,
- if $d \in (1/4, 1/3]$, then W has three components C_1, C_2 and C_3 of measures $|C_1| = |C_2| = d$ and $|C_3| = 1 - 2d$, W is equal to one almost everywhere on $C_1^2 \cup C_2^2$ and $d(\bullet\bullet, W[C_3]) = 0$, and
- if $d = 1/4$, then graphon W has four components C_1, \dots, C_4 , each of the components has measure $1/4$, and W is equal to one almost everywhere on $C_1^2 \cup C_2^2 \cup C_3^2 \cup C_4^2$.

4 Triangle vs. co-cherry projection

In this section, we determine the projection S_{13} . We start with defining several auxiliary functions. The first two functions represent the asymptotic densities of co-cherries and triangles in graphs that have the following structure. Let $\sigma \in (1/4, 1/3]$. The graph have three components: two of the components are complete graphs on the fraction σ of all vertices, and the remaining component consists of two cliques C_1 and C_2 , each formed by the fraction of $(1 - 2\sigma)/2$ of all vertices, such that each vertex of C_i is adjacent to the fraction of

$$\delta_A(\sigma) = \frac{4\sigma - 1}{(1 - 2\sigma)\sqrt{5 - 12\sigma}} \quad (2)$$

of the vertices of C_{3-i} , i.e., each vertex of this component is adjacent to the fraction $(1 + \delta_A(\sigma))\frac{1-2\sigma}{2}$ of all vertices of the graph. An example of such a graph can be found in Figure 3.

The asymptotic co-cherry and triangle densities $h_{A,1}$ and $h_{A,3}$ in graphs having this structure are the following:

$$\begin{aligned} h_{A,1}(\sigma) &= \frac{9 - 48\sigma + 114\sigma^2 - 120\sigma^3 + 3(1 - 2\sigma)(4\sigma - 1)^2\sqrt{5 - 12\sigma}}{10 - 24\sigma} \\ h_{A,3}(\sigma) &= \frac{2 - 18\sigma + 57\sigma^2 - 60\sigma^3}{5 - 12\sigma} \end{aligned}$$

Note that $h_{A,1}(1/4) = 9/16$ and $h_{A,3}(1/4) = 1/16$; these are the asymptotic co-cherry and triangle densities in a graph consisting of four equal size complete graphs, which is the “limit structure” of the above graphs as σ tends to $1/4$. Also note that both $h_{A,1}$ and $h_{A,3}$ are increasing functions of σ in the interval $[1/4, 1/3]$ and their co-domains are $[9/16, 2/3]$ and $[1/16, 1/9]$, respectively.

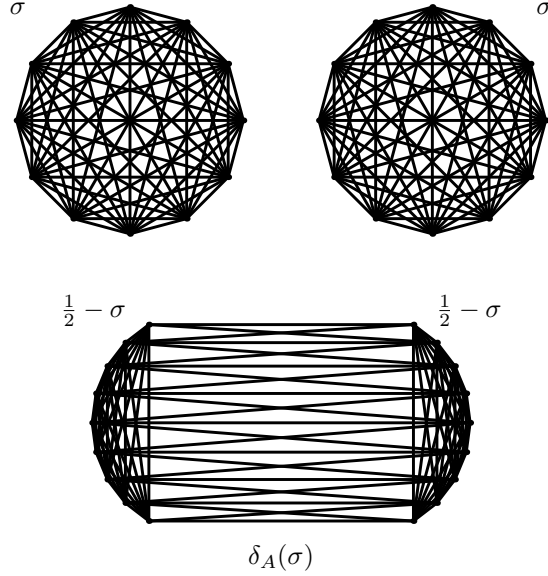


Figure 3: A graph constructed in Section 4, which has co-cherry density $h_{A,1}(\sigma)$ and triangle density $h_{A,3}(\sigma)$, where $\sigma \in (1/4, 1/3]$.

The next two functions $h_{B,1}$ and $h_{B,3}$ are asymptotic co-cherry and triangle densities in a graph consisting of three complete graphs, two formed by σ fraction of all vertices each, and the remaining one formed by $1 - 2\sigma$ of the fraction of all vertices, where $\sigma \in [1/3, 1/2)$.

The functions $h_{B,1}$ and $h_{B,3}$ are defined as follows:

$$\begin{aligned} h_{B,1}(\sigma) &= 6\sigma - 18\sigma^2 + 18\sigma^3 \\ h_{B,3}(\sigma) &= 1 - 6\sigma + 12\sigma^2 - 6\sigma^3 \end{aligned}$$

Note that $h_{A,1}(1/3) = h_{B,1}(1/3) = 2/3$ and $h_{A,3}(1/3) = h_{B,3}(1/3) = 1/9$. In addition, $h_{B,1}(1/2) = 3/4$ and $h_{B,3} = 1/4$, which are the asymptotic co-cherry and triangle densities in a graph formed by two equal size cliques. Also note that both $h_{B,1}$ and $h_{B,3}$ are increasing functions of σ in the interval $[1/3, 1/2]$ with co-domains $[2/3, 3/4]$ and $[1/9, 1/4]$, respectively.

We are now ready to define a function $g_t : [0, 1] \rightarrow [0, 1]$, which will determine the most complex part of the boundary of the projection S_{13} . The function g_t is defined as follows:

$$g_t(x) = \begin{cases} 3x + \frac{3}{8} & \text{if } x \in [0, 1/16], \\ h_{A,1}(h_{A,3}^{-1}(x)) & \text{if } x \in (1/16, 1/9), \\ h_{B,1}(h_{B,3}^{-1}(x)) & \text{if } x \in [1/9, 1/4], \text{ and} \\ 1 - x & \text{otherwise.} \end{cases}$$

We prove in Theorem 12 that the projection S_{13} is equal to the set of the points (d_1, d_3) such that $0 \leq d_1 \leq g_t(d_3)$ and $d_3 \in [0, 1]$. This set is visualized in Figure 4.

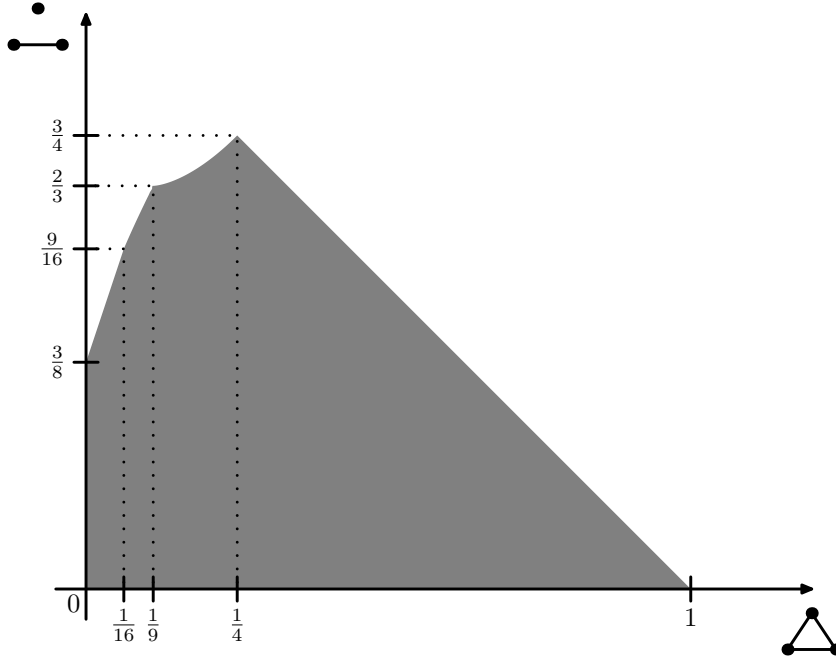


Figure 4: Possible densities of triangles and co-cherries in graphs.

The proof of Theorem 12 is split into several steps. We establish three lemmas that provide different upper bounds on d_1 in terms of d_3 ; each of the upper bounds is tight for a different range of the values d_3 . We start with proving the simplest of the lemmas, which yields the tight upper bound on the initial linear segment of the upper bound on d_1 depicted in Figure 4.

Lemma 6. *Every point (d_1, d_3) contained in S_{13} satisfies $d_1 \leq 3d_3 + 3/8$.*

Proof. The statement of the lemma is equivalent to showing the following inequality in the flag algebra language.

$$\mathbf{\ddot{\cdot}} \leq 3\mathbf{\blacktriangle} + \frac{3}{8}$$

By Proposition 3, we get the following:

$$0 \leq \llbracket (3\mathbf{\blacklozenge} - \mathbf{\ddot{\cdot}})^2 \rrbracket = 9\mathbf{\blacktriangle} + \mathbf{\blacktriangle} - \frac{5}{3}\mathbf{\ddot{\cdot}} + \mathbf{\ddot{\cdot}} \quad (3)$$

Since $1 = \mathbf{\ddot{\cdot}} + \mathbf{\ddot{\cdot}} + \mathbf{\blacktriangle} + \mathbf{\blacktriangle}$, it holds that

$$0 \leq 8\mathbf{\blacktriangle} - \frac{8}{3}\mathbf{\ddot{\cdot}} + 1,$$

which yields the desired inequality $\mathbf{\ddot{\cdot}} \leq 3\mathbf{\blacktriangle} + 3/8$. \square

In the following two subsections, we show that the function $g_t(d_3)$ provides the tight upper bound on d_1 for $d_3 \in (1/16, 1/9)$ and $d_3 \in [1/9, 1/4)$, respectively.

4.1 The concave regime of g_t

In this subsection, we bound d_1 in terms of d_3 for $d_3 \in (1/16, 1/9)$. Before doing so, we need to study the structure of graphons maximizing a certain linear combination of 3-vertex graphs. We start with a lemma saying that almost any two adjacent vertices in such graphons have the same degree.

Lemma 7. *Let $\alpha \in [1, 3]$, and let W be a graphon maximizing $d(\bullet\!\!\!\circ - \alpha \blacktriangle, W)$. It holds for almost every pair $(x, y) \in [0, 1]^2$ with $W(x, y) > 0$ that $d_W(x) = d_W(y)$.*

Proof. The statement of the lemma is equivalent to establishing the following equality.

$$\llbracket ((\mathcal{L} + \mathcal{R}) - (\mathcal{S} + \mathcal{B}))^2 \rrbracket = \llbracket (\mathcal{L} - \mathcal{S})^2 \rrbracket = 0 \quad (4)$$

Indeed, if x and y are the two root vertices in (4), then the expression on the left side of (4) is equal to $\llbracket (d_W(x) - d_W(y))^2 \rrbracket$. This implies that the left side of (4) is equal to

$$\int_{(x,y) \in [0,1]^2, W(x,y) > 0} (d_W(x) - d_W(y))^2 \, dx \, dy;$$

this integral is zero if and only if the assertion of the lemma holds. Hence, we need to prove (4).

We now derive several inequalities using the differential method described in [25, Subsection 4.3] that must be satisfied by any graphon W maximizing $d(\bullet\!\!\!\circ - \alpha \blacktriangle, W)$. The first inequality corresponds to deleting an edge between two distinguished vertices, and it directly follows from [25, Theorem 4.5]. The inequality (5) holds for almost every choice of an edge.

$$0 \geq \partial_{\infty}(\bullet\!\!\!\circ - \alpha \blacktriangle) = 3(\mathcal{L} + \mathcal{S} - \mathcal{O} + \alpha \mathcal{B}) \quad (5)$$

Using (5), we derive that

$$\begin{aligned} 0 \leq 6 \llbracket (\mathcal{O} - \mathcal{L} - \mathcal{S} - \alpha \mathcal{B}) \times (\mathcal{L} + \mathcal{S}) \rrbracket = \\ \mathcal{L}\mathcal{L} - 3\mathcal{L}\mathcal{S} - (1 + \alpha)\mathcal{L}\mathcal{B} - 4\mathcal{S}\mathcal{L} - 2\alpha\mathcal{S}\mathcal{B}, \end{aligned} \quad (6)$$

which also follows from [25, Corollary 4.6]. We next consider the operation of adding an edge and we obtain following the lines of reasoning for deleting an edge that the following holds for almost every choice of a non-edge.

$$0 \geq \partial_{\circ\circ}(\bullet\!\!\!\circ - \alpha \blacktriangle) = 3(\mathcal{O} - \mathcal{L} - \mathcal{S} - \alpha \mathcal{B}) \quad (7)$$

Using (7), we get that it holds that

$$\begin{aligned} 0 \leq 12 \llbracket (\mathcal{L} + \mathcal{S} - \mathcal{O} + \alpha \mathcal{B}) \times \mathcal{B} \rrbracket = \\ -\mathcal{L}\mathcal{B} - 3\mathcal{L}\mathcal{S} + 2\mathcal{L}\mathcal{L} + 2\mathcal{S}\mathcal{B} + 4\alpha\mathcal{L}\mathcal{B} + 2\alpha\mathcal{S}\mathcal{B}. \end{aligned} \quad (8)$$

The final operation that we consider is the following operation: consider a cherry labelled in such a way that the leaves are the first and the third vertices. The

operation that we consider is the operation of removing the edge between the second and third vertices, and adding the edge between the first and the third vertices. Following the way the inequalities (5) and (7) were derived, we obtain that the following holds for almost every choice of a cherry.

$$0 \geq \textcircled{\circ} - \textcircled{\circ} + \alpha \textcircled{\circ} - \alpha \textcircled{\circ}$$

Hence, it holds that

$$0 \leq 12 \llbracket \textcircled{\circ} - \textcircled{\circ} + \alpha \textcircled{\circ} - \alpha \textcircled{\circ} \rrbracket = -3 \textcircled{\circ} + \textcircled{\circ} - \alpha \textcircled{\circ} + 4\alpha \textcircled{\circ}. \quad (9)$$

We get the following inequality by summing the inequality (6) multiplied by $\frac{\alpha-1}{\alpha+1}$, (8) multiplied by $\frac{\alpha-1}{\alpha+1}$, and (9) multiplied by $\frac{3-\alpha}{\alpha+1}$.

$$0 \leq -3 \textcircled{\circ} + \textcircled{\circ} - \textcircled{\circ} + 4 \textcircled{\circ} \quad (10)$$

Since it holds that

$$\llbracket (\textcircled{\circ} - \textcircled{\circ})^2 \rrbracket = \frac{1}{2} \textcircled{\circ} - \frac{1}{6} \textcircled{\circ} + \frac{1}{6} \textcircled{\circ} - \frac{2}{3} \textcircled{\circ},$$

which is a multiple of $-1/6$ of the right side of (10), the left side of (4) cannot be positive. Since the left side of (4) is non-negative by Proposition 3, the equality (4) now follows, which finishes the proof of the lemma. \square

The next lemma concerns degrees of non-adjacent vertices.

Lemma 8. *Let $\alpha \in [1, 3)$, and let W be a graphon maximizing $d(\textcircled{\circ} - \alpha \textcircled{\circ}, W)$. Then for almost every pair $(x, y) \in [0, 1]^2$ with $W(x, y) < 1$ the inequality $d_W(x) + d_W(y) \geq 1/2$ holds.*

Proof. Following the reasoning in the proof of Lemma 7, we get that almost every choice of a non-edge, i.e., almost every pair $(x, y) \in [0, 1]^2$ with $W(x, y) < 1$, satisfies (7). Hence, it holds that

$$0 \leq \alpha \textcircled{\circ} + \textcircled{\circ} + \textcircled{\circ} - \textcircled{\circ} = (\alpha + 1) \textcircled{\circ} + 2 \textcircled{\circ} + 2 \textcircled{\circ} - 1 \leq 4 \textcircled{\circ} + 2 \textcircled{\circ} + 2 \textcircled{\circ} - 1$$

for almost every choice of non-edge. Since $\textcircled{\circ} + \textcircled{\circ} + 2 \textcircled{\circ}$ is equal to $d_W(x) + d_W(y)$, the lemma now follows. \square

To get the desired bound on the co-cherry density, we analyze an optimization problem, which involves the derivative of the function g_t in the interval $(1/16, 1/9)$. Before stating the lemma, it is useful to compute this derivative. We start by investigating the derivatives of the functions $h_{A,1}$ and $h_{A,3}$:

$$\begin{aligned} h'_{A,1}(\sigma) &= \frac{6(4\sigma - 1)(60\sigma^2 - 51\sigma + 11)(1 + \sqrt{5 - 12\sigma})}{(5 - 12\sigma)^2} \\ h'_{A,3}(\sigma) &= \frac{6(4\sigma - 1)(60\sigma^2 - 51\sigma + 11)}{(5 - 12\sigma)^2} \end{aligned}$$

It now follows that the derivative of the function g_t at a point $x \in (1/16, 1/9)$ is the following:

$$g'_t(x) = 1 + \sqrt{5 - 12h_{A,3}^{-1}(x)}.$$

In particular, g'_t is decreasing in the interval $(1/16, 1/9)$ and $2 < g'_t(x) < 1 + \sqrt{2}$.

Lemma 9. *Let $x \in (1/16, 1/9)$. The following inequality holds for every point $(d_1, d_3) \in S_{13}$:*

$$d_1 - g'_t(x)d_3 \leq g_t(x) - g'_t(x)x.$$

In particular, $d_1 \leq g_t(d_3)$ for every point $(d_1, d_3) \in S_{13}$ with $d_3 \in (1/16, 1/9)$.

Proof. Fix $x \in (1/16, 1/9)$ for the proof. We need to show that if W is a graphon maximizing $d(\bullet\bullet - g'_t(x)\blacktriangle, W)$, then $d(\bullet\bullet - g'_t(x)\blacktriangle, W) \leq g_t(x) - g'_t(x)x$. Set $\alpha = g'_t(x)$ and fix a graphon W_0 maximizing $d(\bullet\bullet - \alpha\blacktriangle, W)$. Further, let $(C_j)_{j \in J}$ be the set of components of graphon W_0 with positive measure.

Let $D = [0, 1] \setminus \cup_{j \in J} C_j$. Observe that W_0 is zero almost everywhere on $D \times [0, 1]$. In particular, $d_{W_0}(x) = 0$ for almost every $x \in D$. Hence, Lemma 8 implies that D is null. Consequently, we can assume that $\cup_{j \in J} C_j$ is equal to $[0, 1]$ since the set can be added to one of the components in the collection $(C_j)_{j \in J}$ without violating the constraints.

Suppose that $|J| \neq 1$. Since $d_{W_0}(x) \leq |C_j|$ for every $j \in J$ and almost every $x \in C_j$, Lemma 8 implies that $|C_j| + |C_{j'}| \geq 1/2$ for every distinct $j, j' \in J$. This yields that the number of components is at most four; otherwise, there would be two components with the sum of their measures less than $1/2$.

If W_0 has exactly four components, then Lemma 8 gives that they are of equal size and each of them is a clique. For such W_0 we have $d_3 = 1/16$ and $d_1 = g_t(1/16)$, then since g_t is continuous in $[1/16, 1/9]$ and g'_t is decreasing in $(1/16, 1/9)$, we have $g_t(x) \geq d_1 + g'_t(x)(x - d_3)$. So W_0 fulfills the desired inequality.

Hence, we can assume that $J = \{1, \dots, |J|\}$ and $|J|$ is one, two or three.

Lemma 7 implies that there exist δ_j , $j \in J$, such that $d_{W_0}(x) = \delta_j|C_j|$ for almost every $x \in C_j$. Otherwise, there would exist a real δ' , a partition of C_j to two non-null sets C'' and C''' such that $d_{W_0}(x) \leq \delta'$ for almost every $x \in C''$ and $d_{W_0}(x) > \delta'$ for almost every $x \in C'''$. Since W_0 is not zero almost everywhere on $C' \times C'''$, Lemma 7 would yield a contradiction.

Let $t_j = d(\blacktriangle, W_0[C_j])$ for $j \in J$. Observe that

$$\blacktriangle = 3 \llbracket \textcolor{brown}{\text{J}}^2 \rrbracket - 3\blacktriangle, \bullet\bullet = 3\textcolor{brown}{\text{J}} - 2\blacktriangle - 3\blacktriangle \text{ and } \bullet\bullet = 1 - \blacktriangle - \blacktriangle - \bullet\bullet.$$

Since $d_{W_0}(x) = \delta_j|C_j|$ for almost every $x \in C_j$, it follows that

$$\begin{aligned} d(\blacktriangle, W_0[C_j]) &= t_j \\ d(\blacktriangle, W_0[C_j]) &= 3\delta_j^2 - 3t_j \\ d(\bullet\bullet, W_0[C_j]) &= 3\delta_j - 6\delta_j^2 + 3t_j \\ d(\bullet\bullet, W_0[C_j]) &= 1 - 3\delta_j + 3\delta_j^2 - t_j \end{aligned} \tag{11}$$

In addition, note that $d(\pmb{\mathscr{I}}, W_0[C_j]) = \delta_j$. It now follows that

$$d(\pmb{\mathscr{A}}, W_0) = \sum_{j \in J} t_j |C_j|^3 \quad (12)$$

$$d(\pmb{\mathscr{B}}, W_0) = \sum_{j \in J} (3\delta_j - 6\delta_j^2 + 3t_j) |C_j|^3 + 3\delta_j |C_j|^2 (1 - |C_j|) \quad (13)$$

The equalities (12) and (13) yield that

$$d(\pmb{\mathscr{B}} - \alpha \pmb{\mathscr{A}}, W_0) = K + \sum_{j \in J} (3 - \alpha) d(\pmb{\mathscr{A}}, W_0[C_j]) |C_j|^3, \quad (14)$$

where K depends on δ_j 's and $|C_j|$'s only.

Fix $j \in J$ for this and the next paragraph. We now show that the graphon $W_0[C_j]$ minimizes $d(\pmb{\mathscr{B}}, W)$ among all graphons W such that $d_W(x) = \delta_j$ for almost every $x \in [0, 1]$. Suppose that there exists a graphon W such that $d(\pmb{\mathscr{B}}, W) < d(\pmb{\mathscr{B}}, W_0[C_j])$ and $d_W(x) = \delta_j$ for almost every $x \in [0, 1]$, and consider the graphon W' that is equal to W_0 everywhere outside C_j^2 and that satisfies $W'[C_j] = W$. By (11), it holds that $d(\pmb{\mathscr{A}}, W_0[C_j]) < d(\pmb{\mathscr{A}}, W'[C_j])$, and so $d(\pmb{\mathscr{B}} - \alpha \pmb{\mathscr{A}}, W_0) < d(\pmb{\mathscr{B}} - \alpha \pmb{\mathscr{A}}, W')$ by (14) since $\alpha \in (2, 1 + \sqrt{2})$. However, this contradicts the choice of W_0 .

We claim that $\delta_j > 1/2$ and $d(\pmb{\mathscr{B}}, W_0[C_j]) = 0$. If $W_0[C_j]$ is equal to one almost everywhere on C_j^2 , then $\delta_j = 1$ and the claim follows. Otherwise, we can apply Lemmas 7 and 8 to two vertices of C_j and conclude that $2\delta_j |C_j| \geq 1/2$, which yields that $\delta_j \geq 1/4$. Since every graphon W that minimizes $d(\pmb{\mathscr{B}}, W)$ among all graphons W such that $d_W(x) \leq \delta_j$ satisfies that $d_W(x) = \delta_j$ for almost every $x \in [0, 1]$ by Theorem 5, it follows that $W_0[C_j]$ is one of the graphons listed in the statement of Theorem 5. Since the graphon $W_0[C_j]$ has a single component, it follows that $\delta_j > 1/2$ and $d(\pmb{\mathscr{B}}, W_0[C_j]) = 0$.

Using (11), (12) and (13), we conclude that the following equalities hold.

$$\begin{aligned} d(\pmb{\mathscr{A}}, W_0) &= \sum_{j \in J} (1 - 3\delta_j + 3\delta_j^2) |C_j|^3 \\ d(\pmb{\mathscr{B}}, W_0) &= \sum_{j \in J} (3 - 9\delta_j + 3\delta_j^2) |C_j|^3 + 3\delta_j |C_j|^2 \end{aligned}$$

We claim that $d(\pmb{\mathscr{B}} - \alpha \pmb{\mathscr{A}}, W_0)$ is equal to the maximum value of the function

$$F(x_1, x_2, x_3, y_1, y_2, y_3) = \sum_{j=1}^3 x_j^3 (3 - \alpha - 3(3 - \alpha)y_j - 3(\alpha - 1)y_j^2) + 3x_j^2 y_j \quad (15)$$

subject to that

$$x_1, x_2, x_3 \geq 0, \quad (16)$$

$$x_1 + x_2 + x_3 = 1, \text{ and} \quad (17)$$

$$y_1, y_2, y_3 \in (1/2, 1]. \quad (18)$$

Indeed, setting $x_j = |C_j|$, $y_j = \delta_j$ for $j \in J$, and setting $x_j = 0$, $y_j = 1$ for $j > |J|$ results in a feasible solution, and the value of (15) is equal to $d(\bullet - \alpha \blacktriangle, W_0)$. On the other hand, suppose that x_1, x_2, x_3 and y_1, y_2, y_3 is a feasible solution. Let C_1, C_2, C_3 be arbitrary disjoint subsets of $[0, 1]$ of measures x_1, x_2, x_3 , respectively, and let W be the graphon such that for every $x_j > 0$, $W[C_j]$ is a graphon such that $d_{W[C_j]}(x) = y_j$ for almost every $x \in [0, 1]$ and $d(\bullet, W[C_j]) = 0$; for example, $W[C_j]$ can be chosen as the graphon equal to 1 on $[0, 1/2)^2 \cup [1/2, 1]^2$ and equal to $2y_j - 1$ elsewhere. Since $d(\bullet - \alpha \blacktriangle, W)$ is equal to (15), the choice of W_0 implies that the value of (15) does not exceed $d(\bullet - \alpha \blacktriangle, W_0)$.

The maximum value of (15) subject to (16), (17) and (18) is determined in Proposition 10, and this value is indeed equal to $g_t(x) - \alpha \cdot x$, in particular, it is equal to

$$\begin{aligned} h_{A,1}(\sigma) - \alpha \cdot h_{A,3}(\sigma) &= \frac{24\sigma^3 - 18\sigma^2 + 6\sigma + \sqrt{5 - 12\sigma} - 1}{2\sqrt{5 - 12\sigma}} \\ &= \frac{-\alpha^6 + 6\alpha^5 - 9\alpha^4 - 4\alpha^3 + 96\alpha - 80}{144(\alpha - 1)}, \end{aligned}$$

where $\sigma = \frac{5 - (\alpha - 1)^2}{12}$. □

To complete the proof of Lemma 9, we need to determine the maximum value of (15).

Proposition 10. *Let $\alpha \in (2, 1 + \sqrt{2})$. The maximum value of (15) subject to (16), (17) and (18) is*

$$\frac{-\alpha^6 + 6\alpha^5 - 9\alpha^4 - 4\alpha^3 + 96\alpha - 80}{144(\alpha - 1)}. \quad (19)$$

This value is attained in particular for

$$x_1 = x_2 = \sigma, x_3 = 1 - 2\sigma, y_1 = y_2 = 1 \text{ and } y_3 = \frac{1 + \delta_A(\sigma)}{2},$$

where $\sigma = \frac{5 - (\alpha - 1)^2}{12} = \frac{-\alpha^2 + 2\alpha + 4}{12} \in (1/4, 1/3)$ and $\delta_A(\sigma)$ is defined in (2).

The proof of Proposition 10 can be obtained using any symbolic mathematical computation program. Nevertheless, we include the proof in the Appendix.

4.2 The convex regime of g_t

In this subsection, we deal with graphs having the triangle density between $1/9$ and $1/4$. In order to bound d_1 as a function of d_3 for $d_3 \in [1/9, 1/4)$, we study a certain optimization problem involving d_0, d_1, d_2 and d_3 . We analyze this problem using the result of Pikhurko and Razborov [23], which characterizes the extremal

configurations for the edge vs. triangle density problem. In particular, we show that the optimal values for the problem correspond to 3-vertex graph densities in the complements of the extremal 3-partite complete graphs.

Lemma 11. *Let (d_1, d_3) be a point contained in S_{13} . If $d_3 \in [1/9, 1/4)$, then $d_1 \leq g_t(d_3)$.*

Proof. Let $h(x) : [0, 1] \rightarrow [0, 1]$ be the function equal to $g_R(x)$ for $x \in [0, 2/3]$ and to $x(2x - 1)$ for $x \in [2/3, 1]$. Observe that $h(x) \leq g_R(x)$ for every $x \in [0, 1]$. This inequality can be established by a direct computation. A less technical argument is the following: Goodman's bound asserts the asymptotic lower bound $d_e(2d_e - 1)$ on the triangle density in a graph with edge density d_e , which must be smaller than or equal to the tight asymptotic lower bound $g_R(d_e)$.

Fix $d_3 \in [1/9, 1/4)$ and consider the problem to maximize d_1 subject to

$$d_0 + d_1 + d_2 + d_3 = 1 \quad (20)$$

$$h\left(\frac{3d_0 + 2d_1 + d_2}{3}\right) \leq d_0 \quad (21)$$

where $d_0, d_1, d_2 \geq 0$. Let $m(d_3)$ be this maximum.

Let $(d_0, d_1, d_2, d_3) \in S$, and let $(G_n)_{n \in \mathbb{N}}$ be a convergent sequence of graphs with the limit density of H_k equal to d_k , $k = 0, 1, 2, 3$. The values d_0, d_1, d_2 and d_3 satisfy (20) by Proposition 1. By applying Proposition 2 and Theorem 4 to the limit densities of the complements of the graphs G_n , we obtain that the values d_0, d_1, d_2 and d_3 satisfy that

$$g_R\left(\frac{3d_0 + 2d_1 + d_2}{3}\right) \leq d_0.$$

Since $h(x) \leq g_R(x)$ for every $x \in [0, 1]$, it follows that d_0, d_1, d_2 and d_3 also satisfy the inequality (21). Since the values d_0, d_1 and d_2 are non-negative and they satisfy both (20) and (21), it follows that $d_1 \leq m(d_3)$. In the rest of the proof, we will show $m(d_3) = g_t(d_3)$, which will imply the statement of the lemma.

Recall that $d_3 \in [1/9, 1/4)$ is fixed. We first show that $m(d_3) \geq g_t(d_3)$. Since the functions $h_{B,1}$ and $h_{B,3}$ were defined as the asymptotic co-cherry and triangle densities of graphs with a particular structure, there exists a point (d_0, d_1, d_2, d_3) in S with $d_1 = g_t(d_3) = h_{B,1}(h_{B,3}^{-1}(d_3))$. The values d_0, d_1 and d_2 satisfy both (20) and (21). Hence, it holds that $m(d_3) \geq g_t(d_3)$.

We next establish the opposite inequality, i.e., $m(d_3) \leq g_t(d_3)$. Let d_0, d_1 and d_2 be non-negative reals that satisfy $d_1 = m(d_3)$ and the equations (20) and (21), i.e., d_0, d_1 and d_2 form an optimal solution of the considered maximization problem. Suppose first that (21) is not satisfied with equality. If d_0 or d_2 were positive, we could decrease this variable by some $\varepsilon > 0$ and increase d_1 by ε in such a way that (21) still holds. Since $d_1 = m(d_3)$, this cannot be the case

and we get that both d_0 and d_2 are zero. The equality (20) now implies that $d_1 = 1 - d_3 > 3/4$. Hence, we get that $2d_1/3 > 1/2$ and that the left hand side of (21) must be positive, which is impossible since $d_0 = 0$. We conclude that the values d_0 , d_1 and d_2 satisfy (21) with equality.

We next distinguish three cases depending on the value of d_0 . If $d_0 = 0$, then we get that $d_1 + d_2 = 1 - d_3$ from (20) and that $2d_1/3 + d_2/3 \leq 1/2$ from (21). The maximum d_1 satisfying $d_1 + d_2 = 1 - d_3$ and $2d_1/3 + d_2/3 \leq 1/2$ is equal to $d_3 + 1/2$. Since $d_3 + 1/2 < g_t(d_3)$ for $d_3 \in [1/9, 1/4)$, the triple d_0 , d_1 and d_2 cannot form an optimal solution. The inequality $d_3 + 1/2 < g_t(d_3)$ can be derived as follows. The derivatives of the functions $h_{B,1}$ and $h_{B,3}$ are the following:

$$\begin{aligned} h'_{B,1}(\sigma) &= 6 - 36\sigma + 54\sigma^2 \\ h'_{B,3}(\sigma) &= -6 + 24\sigma - 18\sigma^2 \end{aligned}$$

This implies that the derivative of $g_t(x)$ at a point $x \in (1/9, 1/4)$ is equal to

$$g'_t(x) = \frac{6 - 36\sigma + 54\sigma^2}{-6 + 24\sigma - 18\sigma^2} = 1 - \frac{2 - 4\sigma}{1 - \sigma} ,$$

where $\sigma = h_{B,3}^{-1}(x)$. Since $\sigma \in (1/3, 1/2)$, it follows that $g'_t(x) < 1$ for every $x \in (1/9, 1/4)$. Finally, since the function $g_t(x)$ is continuous and for $x = 1/4$ we have $g_t(x) = x + 1/2$, we get that $g_t(x) > x + 1/2$ for every $x \in [1/9, 1/4)$.

The next case that we analyze is that $d_0 \in (0, 2/9]$. Since the inequality (21) holds with equality, we get that

$$d_0 + \frac{2d_1}{3} + \frac{d_2}{3} = h^{-1}(d_0) ,$$

where h^{-1} is the inverse of the function h restricted to the interval $[1/2, 1]$. We derive using (20) that

$$d_1 = d_3 - 1 + 3h^{-1}(d_0) - 2d_0 .$$

We now investigate the derivative of the function g_R on the interval $[1/2, 2/3]$, which has values as described in (1) on this interval:

$$g'_R(x) = \left(\frac{(1 - \sqrt{4 - 6x})(2 + \sqrt{4 - 6x})^2}{18} \right)' = 1 + \sqrt{1 - \frac{3x}{2}} .$$

Since $h(x) = g_R(x)$ for $x \in [1/2, 2/3]$, we get that the derivative of $h(x)$ on $(1/2, 2/3)$ is strictly between 1 and $3/2$. Hence, the derivative of $h^{-1}(x)$ on $(0, 2/9)$ is strictly between $2/3$ and 1, which implies that $3h^{-1}(d_0) - 2d_0$ is an increasing function of d_0 on $[0, 2/9]$. Since we are considering a triple d_0 , d_1 and d_2 maximizing d_1 , it must hold that $d_0 = 2/9$ or $d_2 = 0$ (if $d_0 < 2/9$ and $d_2 > 0$, we could decrease d_2 while increasing d_0 and so d_1).

Suppose that $d_2 > 0$. It follows that $d_0 = 2/9$, and we derive from (20) and from $d_3 \geq 1/9$ that $d_1 < 1 - d_0 - d_3 \leq 2/3$. Since $m(d_3) \geq 2/3$, such a triple d_0, d_1 and d_2 cannot form an optimal solution. Hence, it must hold that $d_2 = 0$. Since (21) holds with equality and $d_2 = 0$, the result of Pikhurko and Razborov [23] on the asymptotic structure of graphs with a given edge density that minimize the triangle density (see Section 3) implies that d_0, d_1, d_2 and d_3 are equal to densities in the complement of a graph formed by three cliques, where the two largest cliques have the same size. Since the common size of the two largest cliques uniquely determines d_3 , it follows that $d_1 = h_{B,1}(h_{B,3}^{-1}(d_3)) = g_t(d_3)$.

The final case that remains to be analyzed is that $d_0 > 2/9$. Since (21) holds with equality, we again get that

$$d_0 + \frac{2d_1}{3} + \frac{d_2}{3} = h^{-1}(d_0) ,$$

where h^{-1} is the inverse of the function h restricted to the interval $[1/2, 1]$. We express d_1 and substitute to (20) to get that

$$d_2 = 2 - 2d_3 + d_0 - 3h^{-1}(d_0) . \quad (22)$$

Since the derivative of $h(x)$ on the interval $(2/3, 1)$ is equal to $4x - 1$, i.e., the derivative $h'(x)$ is less than 3 for $x \in (2/3, 1)$, we obtain that $x - 3h^{-1}(x)$ is a strictly decreasing function of $x \in (2/9, 1)$. In particular, it holds that

$$d_0 - 3h^{-1}(d_0) < \frac{2}{9} - 3h^{-1}\left(\frac{2}{9}\right) = -\frac{16}{9} .$$

Since $d_3 \geq 1/9$, we obtain from (22) that

$$d_2 = 2 - 2d_3 + d_0 - 3h^{-1}(d_0) < 2 - \frac{2}{9} - \frac{16}{9} = 0 ,$$

which is impossible. This finishes the analysis of the optimal value $m(d_3)$ and we can now conclude that $m(d_3) = g_t(d_3)$ as desired. \square

4.3 The projection S_{13}

We are now ready to determine the projection S_{13} , which is visualized in Figure 4.

Theorem 12. *The projection S_{13} consists precisely of the points (d_1, d_3) such that $0 \leq d_3 \leq 1$ and $0 \leq d_1 \leq g_t(d_3)$.*

Proof. Let T be the set of the points (d_1, d_3) that satisfy $0 \leq d_3 \leq 1$ and $0 \leq d_1 \leq g_t(d_3)$. We first show that $S_{13} \subseteq T$. Let $(d_1, d_3) \in S_{13}$. If $d_3 \in [0, 1/16]$, then $d_1 \leq g_t(d_3) = 3d_3 + 3/8$ by Lemma 6. If $d_3 \in (1/16, 1/9)$, then $d_1 \leq g_t(d_3)$ by Lemma 9. Finally, if $d_3 \in [1/9, 1/4]$, then $d_1 \leq g_t(d_3)$ by Lemma 11, and if

$d_3 \in [1/4, 1]$, then $d_1 \leq 1 - d_3 = g_t(d_3)$ by Proposition 1. We conclude that S_{13} is a subset of T .

We will next show that two particular sets T_1 and T_2 are subsets of S_{13} . The set $T_1 \subseteq \mathbb{R}^2$ is the union of the segments with end-points $(0, x)$ and $(g_t(x), x)$ for $x \in [0, 1/4]$; the set $T_2 \subseteq \mathbb{R}^2$ is the convex hull of the points $(0, 1/4)$, $(3/4, 1/4)$ and $(0, 1)$. Note that $T_1 \cup T_2 = T$.

Let $G_0(n, x)$ for every $x \in [-1/4, 1/4]$ be the following n -vertex graph. If $x \in [-1/4, 0)$, then the vertices of $G_0(n, x)$ are split into five parts A , B , C , D and E such that $|A| = |B| = |C| = |D| = \lfloor (1/4 + x)n \rfloor$ and the remaining vertices belong to E . The graph $G_0(n, x)$ contains all edges between the sets A and B , all edges between the sets C and D , and no other edges.

If $x \in [0, 1/16)$, then the graph $G_0(n, x)$ is the following random n -vertex graph. Its vertices are split into four parts A , B , C and D such that the sizes of any two of the parts differ by at most one. Two vertices inside the same part are joined by an edge with probability $16x$. A pair of vertices from the parts A and B , respectively, is joined by an edge with probability $1 - 16x$; likewise, a pair of vertices from the parts C and D , respectively, is joined by an edge with probability $1 - 16x$.

If $x \in [1/16, 1/9)$, let $\sigma = h_{A,3}^{-1}(x)$. The graph $G_0(n, x)$ has the vertices split into four parts A , B , C and D such that $|A| = |B| = \lfloor (1 - \sigma)n/2 \rfloor$ and the remaining vertices are split among C and D in such a way that the sizes of C and D differ by at most one. Each vertex of A is adjacent to exactly $\lfloor \delta_A(\sigma)|B| \rfloor$ vertices of B , and each vertex of B is adjacent to exactly $\lfloor \delta_A(\sigma)|A| \rfloor$ vertices of A , where $\delta_A(\sigma)$ is as in (2); since $\delta_A(\sigma) \in [0, 1]$ and $|A| = |B|$, this is possible. In addition, all pairs of vertices inside each of the four parts are joined by edges.

Finally, if $x \in [1/9, 1/4]$, then let $\sigma = h_{B,3}^{-1}(x)$. The graph $G_0(n, x)$ has the vertices split into three parts A , B and C such that $|A| = |B| = \lfloor \sigma n \rfloor$, and two vertices are joined by an edge if they belong to the same part.

For every $x \in [-1/4, 1/4]$, the sequence $(G_0(n, x))_{n \in \mathbb{N}}$ is convergent with probability one and the limit triangle density is $\max\{0, x\}$, and the limit co-cherry density is $24 \left(\frac{1}{4} + x\right)^2 \left(\frac{1}{4} - x\right)$ if $x \in [-1/4, 0)$, and it is equal to $g_t(x)$ if $x \in [0, 1/4]$.

We next define a graph $G_1(n, a, x)$ for $a \in [0, 1]$ and $x \in [-1/4, 1/4]$ to be the graph obtained from the graph $G_0(\lceil (1 - a)n \rceil, x)$ by adding $\lfloor an \rfloor$ vertices that are adjacent to all vertices of the graph $G_1(n, a, x)$. The sequence of graphs $(G_1(n, a, x))_{n \in \mathbb{N}}$ is convergent with probability one for every $a \in [0, 1]$ and $x \in [-1/4, 1/4]$; let $h_1(a, x) \in \mathbb{R}^2$ be the pair formed by the limit co-cherry density and the limit triangle density.

Let $t_1 \subseteq \mathbb{R}^2$ be the convex hull of the points $(0, -1/4)$, $(0, 1/4)$, $(1, 1/4)$ and

$(1, -1/4)$. Observe that the following holds for all $a \in [0, 1]$ and $x \in [0, 1/4]$:

$$\begin{aligned}
h_1(0, -x) &= \left(24 \left(\frac{1}{4} + x \right)^2 \left(\frac{1}{4} - x \right), 0 \right) \\
h_1(0, x) &= (g_t(x), x) \\
h_1(a, 1/4) &= \left(\frac{3}{4}(1-a)^3, 1 - \frac{3}{4}(1+a)(1-a)^2 \right) \\
h_1(1, x) &= (0, 1) \\
h_1(1, -x) &= (0, 1) \\
h_1(a, -1/4) &= (0, a^3 + 3a^2(1-a))
\end{aligned}$$

Since the derivative of $1 - \frac{3}{4}(1+a)(1-a)^2$ is equal to

$$\frac{3}{4}(1-a)(1+3a),$$

which is positive for $a \in (0, 1)$, we conclude that the closed region of \mathbb{R}^2 bounded by the image of the boundary of t_1 contains the set T_1 . Since h_1 is a continuous map from t_1 to \mathbb{R}^2 and t_1 is a topological 2-disc, it follows that $T_1 \subseteq h_1(t_1)$. Finally, since $h_1(t_1)$ is a subset of S_{13} , T_1 is also a subset of S_{13} .

It remains to show that T_2 is also a subset of S_{13} . Define $G_2(n, a, p)$ to be the random n -vertex graph that is obtained as we now describe. Split the vertices into two sets A and B with $\lfloor an \rfloor$ and $\lceil (1-a)n \rceil$ vertices, respectively. Join all vertices in the set A by edges, join two vertices in different sets by an edge with probability p , and join two vertices in the set B by an edge with probability $1-p$. The sequence $(G_2(n, a, p))_{n \in \mathbb{N}}$ converges with probability one for all $a, p \in [0, 1]$. The following are the limit co-cherry and triangle densities in this sequence.

$$\begin{aligned}
&3(1-a)^3 p^2(1-p) + 3a(1-a)^2((1-p)^3 + 2p^2(1-p)) + 3a^2(1-a)(1-p)^2 \\
&(1-a)^3(1-p)^3 + 3a(1-a)^2 p^2(1-p) + 3a^2(1-a)p^2 + a^3
\end{aligned}$$

Let $h_2(a, p) \in \mathbb{R}^2$ be these limit density of co-cherries and triangles.

Let $t_2 \subseteq \mathbb{R}^2$ be the convex hull of the points $(0, 1)$, $(1/2, 0)$, $(1, 0)$ and $(1, 1)$, and observe that the following holds:

$$\begin{aligned}
h_2(a, 1-2a) &= (6a(8a^5 - 18a^4 + 7a^3 + 7a^2 - 5a + 1), \\
&\quad a^2(16a^4 - 60a^3 + 78a^2 - 42a + 9)) \\
h_2(a, 0) &= (3a(1-a), 1 - 3a(1-a)) \\
h_2(1, p) &= (0, 1) \\
h_2(a, 1) &= (0, a^2(3-2a))
\end{aligned}$$

Since the derivative of $a^2(16a^4 - 60a^3 + 78a^2 - 42a + 9)$ is equal to

$$6a(1-a)(3 - 18a + 34a^2 - 16a^3),$$

which is positive for $a \in (0, 1/2)$, we conclude that the closed region of \mathbb{R}^2 bounded by the image of the boundary of t_2 contains T_2 . Since h_2 is a continuous map from t_2 to \mathbb{R}^2 and t_2 is a topological 2-disc, it follows that $T_2 \subseteq h_2(t_2)$. Finally, since $h_2(t_2)$ is a subset of S_{13} , T_2 is a subset of S_{13} , which finishes the proof of the theorem. \square

5 Cherry vs. co-cherry projection

In this section, we determine the projection S_{12} , which turned out to be the easiest among the three projections S_{12} , S_{13} and S_{23} . The projection is visualized in Figure 5.

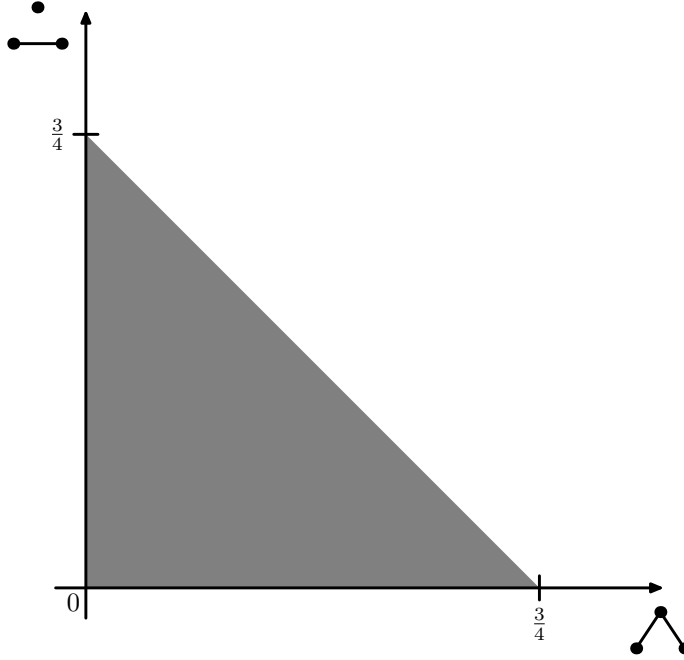


Figure 5: Possible densities of cherries and co-cherries in graphs.

Theorem 13. *The projection S_{12} consists precisely of the points (d_1, d_2) such that $d_1 \geq 0$, $d_2 \geq 0$ and $d_1 + d_2 \leq 3/4$.*

Proof. Let T be the set of the points (d_1, d_2) that satisfy $d_1 \geq 0$, $d_2 \geq 0$ and $d_1 + d_2 \leq 3/4$. We aim to show that $S_{12} = T$. Consider a point $(d_0, d_1, d_2, d_3) \in S$. Since $d_0 + d_3 \geq 1/4$ by Goodman's bound, it follows from Proposition 1 that $d_1 + d_2 \leq 3/4$. Hence, the projection S_{12} is a subset of the set T .

We now define an n -vertex random graph $G(n, a, p)$ where $a, p \in [0, 1]$ as follows: split the vertices of G into a set A of size $\lfloor an \rfloor$ and a set B of size $\lceil (1 - a)n \rceil$, a pair of vertices inside the set A or inside the set B is joined by

an edge with probability p , and the remaining pairs of vertices are joined with probability $1 - p$. The sequence of random graphs $(G(n, a, p))_{n \in \mathbb{N}}$ is convergent with probability one. Observe that the expected co-cherry density in $G(n, a, p)$ is equal to

$$\begin{aligned} (a^3 + (1 - a)^3) \cdot 3p(1 - p)^2 + (3a(1 - a)) \cdot (p^3 + 2(1 - p)^2p) = \\ 3p(1 - p)^2 + 3a(1 - a)p(2p - 1) \end{aligned} \quad (23)$$

and the expected cherry density is equal to

$$\begin{aligned} (a^3 + (1 - a)^3) \cdot 3p^2(1 - p) + (3a(1 - a)) \cdot ((1 - p)^3 + 2p^2(1 - p)) = \\ 3p^2(1 - p) + 3a(1 - a)(1 - p)(1 - 2p). \end{aligned} \quad (24)$$

Let $h(a, p) \in \mathbb{R}^2$ be the limit co-cherry and cherry densities in the sequence $(G(n, a, p))_{n \in \mathbb{N}}$; the standard concentration arguments yield that the sequence converges and the coordinates of $h(a, p)$ are equal to (23) and (24) with probability one, respectively. Note that $h(a, p) \in S_{12}$ for all choices $a, p \in [0, 1]$.

Let T_1 be the subset of T formed by the points $(d_1, d_2) \in T$ with $d_1 \geq d_2$, and let T_2 be the subset formed by the points $(d_1, d_2) \in T$ with $d_1 \leq d_2$. Further, let $t_1 \subseteq \mathbb{R}^2$ be the set of points (x, y) such that $0 \leq x \leq 1/2$ and $0 \leq y \leq x$, and let $t_2 \subseteq \mathbb{R}^2$ be the set of points (x, y) such that $1/2 \leq x \leq 1$ and $x \leq y \leq 1$. Note that t_1 is the convex hull of the points $(0, 0)$, $(1/2, 0)$ and $(1/2, 1/2)$, and t_2 is the convex hull of the points $(1/2, 1/2)$, $(1/2, 1)$ and $(1, 1)$. Observe that the following holds:

$$\begin{aligned} h(a, 0) &= (0, 3a(1 - a)) \\ h(1/2, p) &= \left(\frac{3}{8} - \frac{3}{8}(1 - 2p)^3, \frac{3}{8} + \frac{3}{8}(1 - 2p)^3 \right) \\ h(a, a) &= (3a(1 - a)(1 - 2a + 2a^2), 3a(1 - a)(1 - 2a + 2a^2)) \end{aligned}$$

Hence, the boundary of the triangle t_1 is mapped by h to the boundary of T_1 . Since h is a continuous map from t_1 to \mathbb{R}^2 , t_1 is a topological 2-disc and its boundary is mapped to the boundary of T_1 , it follows $T_1 \subseteq h(t_1)$. Since $h(t_1)$ is a subset of S_{12} , it follows that $T_1 \subseteq S_{12}$. The analogous argument yields that the boundary of triangle t_2 is mapped by h to the boundary of T_2 , which implies that $T_2 \subseteq h(t_2)$ and thus $T_2 \subseteq S_{12}$. We conclude that all the points of $T = T_1 \cup T_2$ are contained in S_{12} , which finishes the proof that $S_{12} = T$. \square

6 Triangle vs. cherry projection

In this section, we determine the last remaining projection S_{23} . Recall that $g_R^{-1} : [0, 1] \rightarrow [1/2, 1]$ is the inverse of the function g_R from Theorem 4 restricted to the interval $[1/2, 1]$.

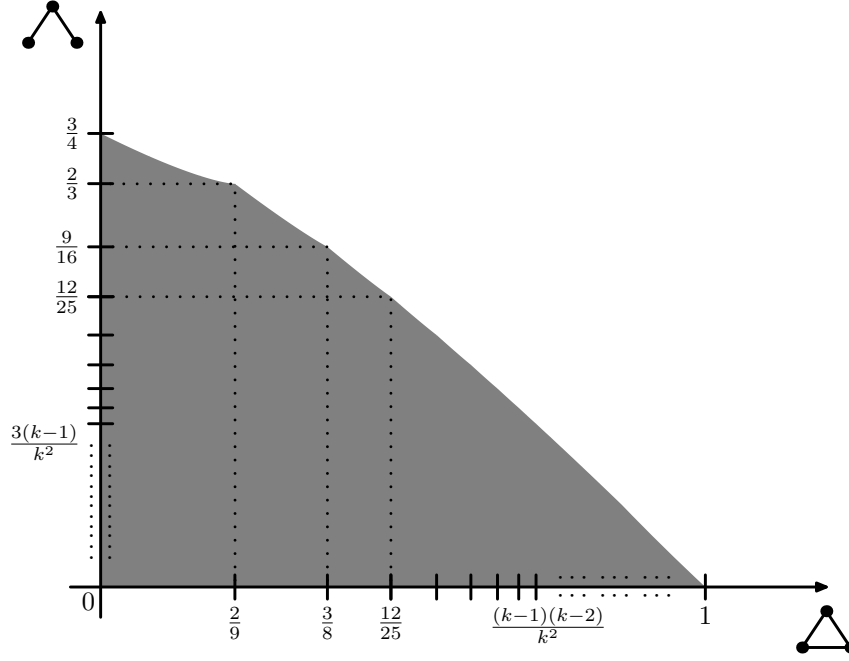


Figure 6: Possible densities of triangles and cherries in graphs.

Theorem 14. *The projection S_{23} consists precisely of the points (d_2, d_3) such that $d_2 \geq 0$, $d_3 \geq 0$ and $d_2 \leq \frac{3}{2} (g_R^{-1}(d_3) - d_3)$.*

Proof. Let T be the set of the points (d_2, d_3) that satisfy the inequalities $d_2 \geq 0$, $d_3 \geq 0$ and $d_2 \leq \frac{3}{2} (g_R^{-1}(d_3) - d_3)$. We will show that $S_{23} = T$. Consider a point $(d_0, d_1, d_2, d_3) \in S$. Clearly, d_1, d_2 and d_3 are non-negative. By Theorem 4, we have that

$$d_3 \geq g_R \left(\frac{d_1 + 2d_2 + 3d_3}{3} \right) .$$

It follows that

$$\frac{2d_2 + 3d_3}{3} \leq \frac{d_1 + 2d_2 + 3d_3}{3} \leq g_R^{-1}(d_3) ,$$

which yields that

$$d_2 \leq \frac{3}{2} (g_R^{-1}(d_3) - d_3) .$$

We conclude that the projection S_{23} is a subset of the set T .

We now define an n -vertex graph $G(n, a, b)$ for $a \in [0, 1/2]$ and $b \in [0, 1]$. The graph $G(n, a, b)$ has $\lfloor (1-b)n \rfloor$ isolated vertices. If $a = 0$, the remaining $\lfloor bn \rfloor$ vertices form a complete graph. Otherwise, the remaining $\lfloor bn \rfloor$ vertices form a complete multipartite graph with $\lfloor a^{-1} \rfloor + 1$ parts such that $\lfloor a^{-1} \rfloor$ parts have size $\lfloor abn \rfloor$ and the remaining part has size $\lfloor bn \rfloor - \lfloor a^{-1} \rfloor \cdot \lfloor abn \rfloor$. One or more of the parts of the complete multipartite graph can be empty; this happens if either bn is an integer, a^{-1} is an integer and a^{-1} divides bn , or $abn < 1$. It is

straightforward to show that the sequence of graphs $(G(n, a, b))_{n \in \mathbb{N}}$ converges for every fixed pair $a \in [0, 1/2]$ and $b \in [0, 1]$. Let $h(a, b) \in \mathbb{R}^2$ be the limit cherry and triangle densities in the sequence $(G(n, a, b))_{n \in \mathbb{N}}$. Observe that $h(a, b)$ is a continuous function from $[0, 1/2] \times [0, 1]$ to \mathbb{R}^2 .

We now investigate the function $h(a, b) : [0, 1/2] \times [0, 1] \rightarrow \mathbb{R}^2$. First observe that $h(a, b) = b^3 \cdot h(a, 1)$. We have $h(0, 1) = (0, 1)$ and thus $h(0, b) = (0, b^3)$. Fix $a \in (0, 1/2]$ and consider the sequence of graphs $G(n, a, 1)$. Let d_e be the limit edge density and let d_k be the limit density of H_k , the k -edge 3-vertex graph. Note that $d_1 = 0$ since each of the graphs $G(n, a, 1)$ is a complete multipartite graph. Next observe that the graph $G(n, a, 1)$ is a complete multipartite graph with $\lfloor a^{-1} \rfloor + 1$ parts such that the fraction of the vertices contained in $\lfloor a^{-1} \rfloor$ of its parts converges to a . Hence, it follows that $d_3 = g_R(d_e)$. The construction of graphs $G(n, a, 1)$ and the fact that $a \leq 1/2$, implies that $d_e \geq 1/2$. Since $d_e = (d_1 + 2d_2 + 3d_3)/3$ (see Proposition 2) and $d_1 = 0$, we get the following:

$$g_R^{-1}(d_3) = \frac{2d_2 + 3d_3}{3}.$$

Thus $d_2 = \frac{3}{2} (g_R^{-1}(d_3) - d_3)$. We conclude that the points $h(a, 1)$, $a \in [0, 1/2]$, form the curve of the boundary of T between the points $(0, 1)$ and $(3/4, 0)$.

Consider now the square $t = [0, 1/2] \times [0, 1]$. We claim that the boundary of t is mapped by h to the boundary of T . Indeed, the segment $\{1/2\} \times [0, 1]$ is mapped to the segment $\{0\} \times [0, 3/4]$, the segment $[0, 1/2] \times \{0\}$ is mapped to the point $(0, 0)$, the segment $\{0\} \times [0, 1]$ is mapped to the segment $[0, 1] \times \{0\}$, and the segment $[0, 1/2] \times \{1\}$ is mapped to the remaining part of the boundary of T . Since h is a continuous map from t to \mathbb{R}^2 , t is a topological 2-disc and its boundary is mapped to the boundary of T , it follows $T \subseteq h(t)$. Since $h(t)$ is a subset of S_{23} , we conclude that $T \subseteq S_{23}$, which yields that $T = S_{23}$. \square

7 Boundaries of the projections

In this section, we briefly discuss the structure of graphs on the boundaries of the projections S_{13} , S_{12} and S_{23} . Those on the parts of the boundaries with the zero density of one of the graphs H_1 , H_2 and H_3 can be any graphs with corresponding zero density. While the structure of triangle-free graphs can be very complex, every graph with the zero density of H_2 is a union of cliques; graphs with the zero density of H_1 are then their complements. The situation is more interesting for the other parts of the boundaries of the projections.

Let us start with the projection S_{13} . We will describe the structure of graphons W with the co-cherry density equal to $g_t(d(H_3, W))$. We distinguish four cases based on the triangle density of W ; each case corresponds to one of the smooth parts of the curve $(x, g_t(x))$, $x \in [0, 1]$. If the triangle density belongs to the interval $[0, 1/16]$, then the upper bound follows from Lemma 6. The equality

$d(H_1, W) = g_t(d(H_3, W))$ holds if and only if (3) in the proof of Lemma 6 holds with equality. Hence, a graphon W satisfies $d(H_1, W) = g_t(d(H_3, W))$ and $d(H_3, W) \in [0, 1/16]$ if and only if $d_W(x) = 1/4$ for almost every $x \in [0, 1]$.

If the triangle density belongs to the interval $(1/16, 1/9)$, then the proof of Lemma 9 yields that a graphon W satisfies $d(H_1, W) = g_t(d(H_3, W))$ if and only if it has three components of measures x_1, x_2 and x_3 with relative edge densities y_1, y_2 and y_3 and zero co-triangle densities such that the values of x_1, x_2, x_3, y_1, y_2 and y_3 maximize the sum (15) subject to (16), (17) and (18). Proposition 10 yields that such values are unique up to their permutation. We conclude that a graphon satisfies $d(H_1, W) = g_t(d(H_3, W))$ and $d(H_3, W) \in (1/16, 1/9)$ if and only if it is a limit of the sequence $(G_0(n, x))_{n \in \mathbb{N}}$ for some $x \in [1/16, 1/9]$, where the graph $G_0(n, x)$ is defined as in the proof of Theorem 12.

If the triangle density belongs to the interval $[1/9, 1/4)$, then the proof of Lemma 11 implies that the density of cherries is zero and the density of co-triangles is equal to $g_R(1 - d(K_2, W))$, i.e., the graphon $1 - W$ is one of the graphons minimizing the triangle density for a given edge density. The structure of such graphons was determined by Pikhurko and Razborov [23]. It follows that a graphon W satisfies that $d(H_1, W) = g_t(d(H_3, W))$ and $d(H_3, W) \in [1/9, 1/4)$ if and only if it is a limit of the sequence $(G_0(n, x))_{n \in \mathbb{N}}$ for some $x \in [1/9, 1/4)$, where the graph $G_0(n, x)$ is defined as in the proof of Theorem 12.

Finally, if the triangle density belongs to the interval $[1/4, 1]$, then Proposition 1 gives that the co-triangle and cherry density must be zero, i.e., such graphons corresponds to unions of two complete graphons. Consequently, a graphon W satisfies that $d(H_1, W) = g_t(d(H_3, W))$ and $d(H_3, W) \in [1/4, 1]$ if and only if it is a limit of the sequence $(G_2(n, a, 0))_{n \in \mathbb{N}}$ for some $a \in [0, 1/2]$, where the graph $G_2(n, a, p)$ is defined as in the proof of Theorem 12.

The situation is less complex for the projections S_{12} and S_{23} . The case of the projection S_{12} is quite simple: the structure of extremal configurations for Goodman's bound implies that a graphon W satisfies $d(H_1, W) + d(H_2, W) = 3/4$ if and only if $d_W(x) = 1/2$ for almost every $x \in [0, 1]$. In the case of the projection S_{23} , we need to inspect the proof of Theorem 14. We note that the equality in the last inequality in the statement of the theorem holds if and only if the graphon has zero co-cherry density and it is one of the extremal configurations described by Pikhurko and Razborov in [23]. It follows that a graphon W satisfies that $d(H_2, W) = \frac{3}{2} (g_R^{-1}(d(H_3, W)) - d(H_3, W))$ if and only if either it is equal to one almost everywhere or there exists k and $\alpha \in [\frac{1}{k}, \frac{1}{k-1})$ and W is a limit of a sequence of complete k -partite graphs with $k - 1$ parts containing the fraction α of the vertices.

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Appendix

We rewrite Proposition 10 in a self-contained form and present a formal proof.

Proposition 10. *Let $\alpha \in (2, 1 + \sqrt{2})$. The maximum value of*

$$F(x_1, x_2, x_3, y_1, y_2, y_3) = \sum_{j=1}^3 x_j^3 (3 - \alpha - 3(3 - \alpha)y_j - 3(\alpha - 1)y_j^2) + 3x_j^2 y_j \quad (25)$$

subject to

$$x_1, x_2, x_3 \geq 0, \quad (26)$$

$$x_1 + x_2 + x_3 = 1, \text{ and} \quad (27)$$

$$y_1, y_2, y_3 \in (1/2, 1]. \quad (28)$$

is

$$\frac{-\alpha^6 + 6\alpha^5 - 9\alpha^4 - 4\alpha^3 + 96\alpha - 80}{144(\alpha - 1)}. \quad (29)$$

This value is attained in particular for

$$x_1 = x_2 = \sigma, x_3 = 1 - 2\sigma, y_1 = y_2 = 1 \text{ and } y_3 = \frac{(1 - 2\sigma)\sqrt{5 - 12\sigma} + 4\sigma - 1}{2(1 - 2\sigma)\sqrt{5 - 12\sigma}},$$

where $\sigma = \frac{5 - (\alpha - 1)^2}{12} = \frac{-\alpha^2 + 2\alpha + 4}{12} \in (1/4, 1/3)$.

Proof. We determine the maximum value of (25) when the constraint (28) is replaced with

$$y_1, y_2, y_3 \in [1/2, 1],$$

and show that the maximum value of this relaxed problem is the same. Before proceeding with the proof, note that if $\alpha \in (2, 1 + \sqrt{2})$, then $\sigma \in (1/4, 1/3)$.

Fix $\alpha \in (2, 1 + \sqrt{2})$ and an optimal solution x_1, \dots, x_3 and y_1, \dots, y_3 . By symmetry, we may assume that $x_1 \leq x_2 \leq x_3$. We start by investigating the partial derivative of (25) with respect to y_j , which is equal to

$$x_j^3 (-3(3 - \alpha) - 6(\alpha - 1)y_j) + 3x_j^2. \quad (30)$$

If $x_j \neq 0$ and (30) is zero, then it holds that

$$y_j = \frac{x_j^{-1} - (3 - \alpha)}{2(\alpha - 1)},$$

which belongs to $[1/2, 1]$ if $x_j \in [(1 + \alpha)^{-1}, 1/2]$. If $x_j = 0$, then the value of (25) does not depend on y_j and we can set it arbitrarily. Hence, we conclude that we can assume that the optimal solution that we have fixed satisfies that

$$y_j = \begin{cases} 1 & \text{if } x_j \leq \frac{1}{\alpha + 1}, \\ 1/2 & \text{if } x_j \geq \frac{1}{2}, \text{ and} \\ \frac{x_j^{-1} - (3 - \alpha)}{2(\alpha - 1)} & \text{otherwise.} \end{cases} \quad (31)$$

If $y_j = 1$, then the j -th term in (25) is equal to $3x_j^2 - (\alpha + 3)x_j^3$, and the partial derivative of (25) with respect to x_j is

$$6x_j - 3(\alpha + 3)x_j^2. \quad (32)$$

If $y_j = 1/2$, then the j -th term is $\frac{3}{2}x_j^2 - \frac{\alpha+3}{4}x_j^3$, and the partial derivative of (25) with respect to x_j is

$$3x_j - \frac{3}{4}(\alpha + 3)x_j^2. \quad (33)$$

Finally, if the third case of (31) applies, then the j -th term in (25) is

$$\frac{(3 - \alpha)(\alpha + 5)x_j^3 - 6(3 - \alpha)x_j^2 + 3x_j}{4(\alpha - 1)},$$

and the partial derivative of (25) with respect to x_j is

$$\frac{3(3 - \alpha)(\alpha + 5)x_j^2 - 12(3 - \alpha)x_j + 3}{4(\alpha - 1)}. \quad (34)$$

We next distinguish three cases depending on how many of the variables x_1, \dots, x_3 are equal to zero. If two of the variables x_1, \dots, x_3 are zero, i.e., $x_1 = x_2 = 0$, then $x_3 = 1$. This implies that $y_3 = 1/2$ and the value of (25) is $\frac{3-\alpha}{4}$, which is less than (29).

Suppose that exactly one of the variables is zero, i.e., $x_1 = 0$. To analyze this case, we need to distinguish four cases depending on the value of x_2 . Note that $x_2 \in (0, 1/2]$.

- **The value of x_2 belongs to $(0, \frac{1}{\alpha+1})$.** Note that $x_3 \in (1/2, 1)$. Using the method of Lagrange multipliers, we get that

$$\frac{\partial}{\partial x_2} F = \frac{\partial}{\partial x_3} F.$$

It follows from (32) and (33) that

$$6x_2 - 3(\alpha + 3)x_2^2 = 3x_3 - \frac{3}{4}(\alpha + 3)x_3^2.$$

This would imply that either $x_2 = 1/3$ and $x_3 = 2/3$, or $x_2 = \frac{1-\alpha}{\alpha+3}$ and $x_3 = \frac{2(\alpha+1)}{\alpha+3}$. In either of the cases, x_2 does not belong to the interval $(0, \frac{1}{\alpha+1})$.

- **The value of x_2 is equal $\frac{1}{\alpha+1}$.** It follows that $x_3 = \alpha/(\alpha + 1)$, $y_2 = 1$ and $y_3 = 1/2$. Consequently, the value of (25) is

$$\frac{-\alpha^4 + 3\alpha^3 + 6\alpha^2 + 8\alpha}{4(\alpha + 1)^3},$$

which is less than (29).

- **The value of x_2 belongs to $(\frac{1}{\alpha+1}, 1/2)$.** Applying the method of Lagrange multipliers, we get that

$$\frac{3(3-\alpha)(\alpha+5)x_2^2 - 12(3-\alpha)x_2 + 3}{4(\alpha-1)} = 3x_3 - \frac{3}{4}(\alpha+3)x_3^2.$$

It follows that

$$x_2 = \frac{\alpha^2 - 2\alpha + 2}{6} \text{ and } x_3 = \frac{-\alpha^2 + 2\alpha + 4}{6}.$$

The value of (25) is then the same as (29). However, since $y_3 = 1/2$, the values of the variables do not form an optimal solution of the original problem.

- **The value of x_2 is equal to $1/2$.** It follows that $x_3 = 1/2$, $y_2 = 1/2$ and $y_3 = 1/2$. Hence, the value of (25) is $\frac{9-\alpha}{16}$, which is less than (29).

In the rest of the proof, we assume that all the three variables x_1 , x_2 and x_3 are positive. We start by considering the cases when two of the values of x_1 , x_2 and x_3 are equal to $\frac{1}{\alpha+1}$ or $1/2$. There are only two cases to analyze (assuming that $x_1 \leq x_2 \leq x_3$).

- **It holds that $x_1 = x_2 = \frac{1}{\alpha+1}$ and $x_3 = \frac{\alpha-1}{\alpha+1}$.** It follows that $y_1 = y_2 = 1$ and

$$y_3 = \frac{\alpha^2 - 3\alpha + 4}{2(\alpha-1)^2}.$$

The value of (25) is then equal to

$$\frac{-\alpha^4 + 6\alpha^3 + 3\alpha^2 - 16\alpha + 36}{4(\alpha+1)^3},$$

which is smaller than (29).

- **It holds that $x_1 = \frac{\alpha-1}{2(\alpha+1)}$, $x_2 = \frac{1}{\alpha+1}$ and $x_3 = \frac{1}{2}$.** We get that $y_1 = y_2 = 1$ and $y_3 = 1/2$. This implies that the value of (25) is

$$\frac{-5\alpha^3 + 35\alpha^2 - 11\alpha + 45}{32(\alpha+1)^2},$$

which is also smaller than (29).

We next consider the cases when exactly one of the values of x_1 , x_2 and x_3 is equal to $\frac{1}{\alpha+1}$ or $1/2$. Recall that $x_1 \leq x_2 \leq x_3$.

- **The value of x_1 belongs to $(0, \frac{1}{\alpha+1})$, the value of x_2 is equal to $\frac{1}{\alpha+1}$, and the value of x_3 is smaller than $1/2$.** Using the method of Lagrange multipliers, we get that

$$\frac{\partial}{\partial x_1} F = \frac{\partial}{\partial x_3} F .$$

It follows using (32) and (34) that

$$6x_1 - 3(\alpha + 3)x_1^2 = \frac{3(3 - \alpha)(\alpha + 5)x_3^2 - 12(3 - \alpha)x_3 + 3}{4(\alpha - 1)} ,$$

which implies that

$$\begin{aligned} x_1 &= \frac{-\alpha^4 + 3\alpha^3 + 15\alpha^2 + \alpha - 10}{3(\alpha + 1)^4} \pm \frac{\sqrt{4\alpha^6 - 8\alpha^5 - 39\alpha^4 + 84\alpha^3 + 74\alpha^2 - 196\alpha + 97}}{3(\alpha + 1)^3} \\ x_3 &= \frac{4\alpha^4 + 6\alpha^3 - 6\alpha^2 + 2\alpha + 10}{3(\alpha + 1)^4} \mp \frac{\sqrt{4\alpha^6 - 8\alpha^5 - 39\alpha^4 + 84\alpha^3 + 74\alpha^2 - 196\alpha + 97}}{3(\alpha + 1)^3} \end{aligned}$$

Since $x_3 > 1/2$ for one of the two choices of the sign, we get that

$$\begin{aligned} x_1 &= \frac{-\alpha^4 + 3\alpha^3 + 15\alpha^2 + \alpha - 10}{3(\alpha + 1)^4} + \frac{\sqrt{4\alpha^6 - 8\alpha^5 - 39\alpha^4 + 84\alpha^3 + 74\alpha^2 - 196\alpha + 97}}{3(\alpha + 1)^3} \\ x_3 &= \frac{4\alpha^4 + 6\alpha^3 - 6\alpha^2 + 2\alpha + 10}{3(\alpha + 1)^4} - \frac{\sqrt{4\alpha^6 - 8\alpha^5 - 39\alpha^4 + 84\alpha^3 + 74\alpha^2 - 196\alpha + 97}}{3(\alpha + 1)^3} , \end{aligned}$$

which yields the value of (25) smaller than (29).

- **The value of x_1 belongs to $(0, \frac{1}{\alpha+1})$, the value of x_2 is equal to $\frac{1}{\alpha+1}$, and the value of x_3 is larger than $1/2$.** We derive using the method of Lagrange multipliers that

$$6x_1 - 3(\alpha + 3)x_1^2 = 3x_3 - \frac{3}{4}(\alpha + 3)x_3^2 .$$

Since $x_3 > 1/2$, it follows that

$$\begin{aligned} x_1 &= \frac{-\alpha^2 + \alpha + 4}{\alpha^2 + 4\alpha + 3} \\ x_3 &= \frac{2(\alpha^2 + \alpha - 2)}{\alpha^2 + 4\alpha + 3} \end{aligned}$$

Consequently, the value of (25) is

$$\frac{-\alpha^5 + \alpha^4 + 11\alpha^3 + 3\alpha^2 - 6\alpha + 24}{(\alpha + 1)^2(\alpha + 3)^2} ,$$

which is smaller than (29).

- **The value of x_1 is equal to $\frac{1}{\alpha+1}$, and the remaining values are larger than $\frac{1}{\alpha+1}$.** Note that both x_2 and x_3 must be less than $1/2$. Since the partial derivatives of F with respect to x_2 and x_3 are equal, it follows that $x_2 = x_3$. Hence, it holds that

$$x_2 = x_3 = \frac{\alpha}{2(\alpha+1)}.$$

Consequently, the value of (25) is

$$\frac{-\alpha(\alpha^4 - 10\alpha^3 - 3\alpha^2 - 20\alpha + 20)}{16(\alpha - 1)(\alpha + 1)^3},$$

which is smaller than (29).

- **Both x_1 and x_2 belong to $(0, \frac{1}{\alpha+1})$, and $x_3 = 1/2$.** Since the partial derivatives of F with respect to x_1 and x_2 must be the same, it follows that $x_1 = x_2 = 1/4$. The value of (25) is then equal to $\frac{9-\alpha}{16}$, which is again smaller than (29).
- **The value of x_1 belongs to $(0, \frac{1}{\alpha+1})$, the value of x_2 belongs to $(\frac{1}{\alpha+1}, 1/2)$, and $x_3 = 1/2$.** We derive using the method of Lagrange multipliers that

$$6x_1 - 3(\alpha + 3)x_1^2 = \frac{3(3 - \alpha)(\alpha + 5)x_2^2 - 12(3 - \alpha)x_2 + 3}{4(\alpha - 1)},$$

It follows that

$$\begin{aligned} x_1 &= \frac{-\alpha^2 - 2\sqrt{\alpha^4 - 8\alpha^3 + 23\alpha^2 - 22\alpha + 10} + 10\alpha - 5}{6(\alpha + 1)^2} \\ x_2 &= \frac{2\alpha^2 + \sqrt{\alpha^4 - 8\alpha^3 + 23\alpha^2 - 22\alpha + 10} - 2\alpha + 4}{3(\alpha + 1)^2}, \end{aligned}$$

and the value of (25) yet again smaller than (29).

It remains to consider the cases when none of the values of x_1 , x_2 and x_3 is equal to $\frac{1}{\alpha+1}$ or $1/2$.

- **Both x_1 and x_2 belong to $(0, \frac{1}{\alpha+1})$, and x_3 belongs to $(\frac{1}{\alpha+1}, 1/2)$.** It follows that $x_1 = x_2$ and that

$$6x_1 - 3(\alpha + 3)x_1^2 = \frac{3(3 - \alpha)(\alpha + 5)x_3^2 - 12(3 - \alpha)x_3 + 3}{4(\alpha - 1)}.$$

We get that either $x_1 = x_2 = 1/4$ and $x_3 = 1/2$, which does not meet the description of the case, or

$$x_1 = x_2 = \frac{-\alpha^2 + 2\alpha + 4}{12} \text{ and } x_3 = \frac{\alpha^2 - 2\alpha + 2}{6}.$$

The latter is the solution given in the statement of the proposition.

- **Both x_1 and x_2 belong to $(0, \frac{1}{\alpha+1})$, and x_3 belongs to $(1/2, 1)$.** We get that $x_1 = x_2$ and that

$$6x_1 - 3(\alpha + 3)x_1^2 = 3x_3 - \frac{3}{4}(\alpha + 3)x_3^2.$$

However, the only solution $x_1 = x_2 = 1/4$ and $x_3 = 1/2$ does not meet the case description.

- **The value of x_1 belongs to $(0, \frac{1}{\alpha+1})$, and both x_2 and x_3 belong to $(\frac{1}{\alpha+1}, 1/2)$.** It follows that $x_2 = x_3$ and that

$$6x_1 - 3(\alpha + 3)x_1^2 = \frac{3(3 - \alpha)(\alpha + 5)x_3^2 - 12(3 - \alpha)x_3 + 3}{4(\alpha - 1)}.$$

We get that

$$\begin{aligned} x_1 &= \frac{-\alpha^2 + 18\alpha - 13 \pm 2\sqrt{4\alpha^4 - 24\alpha^3 + 53\alpha^2 - 30\alpha + 1}}{3 \cdot (5\alpha^2 + 10\alpha - 11)} \\ x_3 &= \frac{8\alpha^2 + 6\alpha - 10 \mp \sqrt{4\alpha^4 - 24\alpha^3 + 53\alpha^2 - 30\alpha + 1}}{3 \cdot (5\alpha^2 + 10\alpha - 11)}, \end{aligned}$$

which results in the value of (25) to be smaller than (29).

- **The value of x_1 belongs to $(0, \frac{1}{\alpha+1})$, the value of x_2 belongs to $(\frac{1}{\alpha+1}, 1/2)$, and the value of x_3 belongs to $(1/2, 1)$.** Using the method of the Lagrange multipliers, we get that

$$6x_1 - 3(\alpha + 3)x_1^2 = \frac{3(3 - \alpha)(\alpha + 5)x_2^2 - 12(3 - \alpha)x_2 + 3}{4(\alpha - 1)} = 3x_3 - \frac{3}{4}(\alpha + 3)x_3^2.$$

Hence, $x_3 = 2x_1$, which implies that $x_1 > 1/4$. However, this is impossible since $x_2 > x_1$ and the sum of x_1 , x_2 and x_3 is equal to one.

The proof of the proposition is now finished. Note that the feasible solution from the statement is unique up to a permutation of the values of the variables. \square